Lecture 23

In this lecture, we'll start with an extremely powerful method in whole of mathematics :- group actions.

Recall Cayley's Theorem :- Every group G is isomorphic to a group of permutations.

So if $g \in G \Rightarrow g \in S_n$ for some n and hence g can be viewed as a permutation!

The idea of an action of a group is to take this point of view further.

Let G be a group and $X \neq \phi$ be a set. We'll see two definitions of a group action.

Definition 1 :- An action of G on X is a map

$\alpha : G \times X \longrightarrow X$, and we'll sometimes write

$\alpha(g, x) = g \cdot x$, $g \in G$ and $x \in X$ s.t.

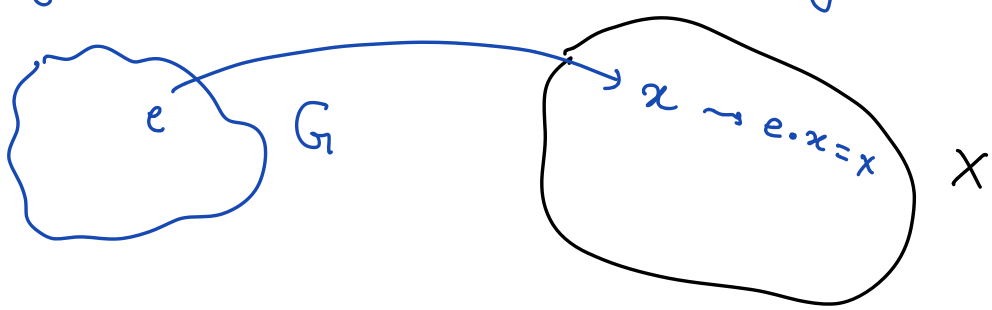
α satisfies the following properties:-

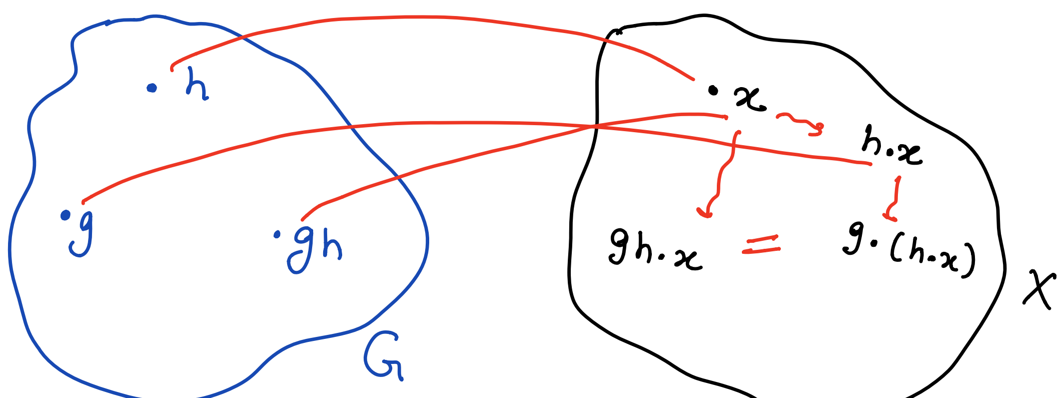
1) $\forall\, x \in X$, $\alpha(e, x) = e \cdot x = x$

2) $\alpha(g, \alpha(h, x)) = \alpha(gh, x)$, i.e.,

$$g \cdot (h \cdot x) = gh \cdot x \quad \forall\, g, h \in G,\ x \in X.$$

So if we see these things pictorially





i.e. if we first look at $h \cdot x$ which will be an element of X and then act by g thus getting $g \cdot (h \cdot x) \in X$ or we act on x by the element $gh \in G$ to get $(gh) \cdot x$, the end result should be the same.

To see the second definition, recall that if X is a set then the

$$P(X) = \{ f : X \to X \mid f \text{ is a bijection} \}$$

is a group under composition of functions.

<u>Definition 2</u> An action of G on X is a homomorphism $\varphi : G \to P(X)$.

So an action is nothing but a homomorphism.

<u>Proposition 1</u> Definition 1) and 2) are equivalent.

<u>Proof</u>:- We will assume Definition 1) and

prove Definition 2) and then vice-versa.

Assume Def^n 1, i.e, we have a map

$\alpha : G \times X \to X$ satisfying the two properties.

We want a homomorphism

$$\varphi : G \to P(X)$$

Define $\varphi(g) = \sigma \in P(X)$ where

$\sigma : X \to X$ is given by $\sigma(x) = \alpha(g, x)$.

Claim 1 σ do belongs to $P(X)$.

If $\sigma(x_1) = \sigma(x_2) \Rightarrow g \cdot x_1 = g \cdot x_2 \;\; \forall g \in G$.

So take $g = e \Rightarrow e \cdot x_1 = x_1 = e \cdot x_2 = x_2$

So $x_1 = x_2 \Rightarrow \sigma$ is one-one.

Also, for $x \in X$, $\alpha(e, x) = \sigma(x) = e \cdot x = x$

$\Rightarrow \sigma$ is onto.

Thus the map $\sigma \in P(X)$.

Claim 2 φ is a homomorphism.

For $g, h \in G$

$\varphi(gh) = \sigma$ where $\sigma(x) = (gh)\cdot x$

But from property 2) of Def^n 1 of an action,

$gh\cdot x = g\cdot(h\cdot x) = \sigma_1 \circ \sigma_2(x)$ where

$\sigma_1(x) = g\cdot x$

$\sigma_2(x) = h\cdot x$

$\Rightarrow \varphi(gh) = \sigma = \sigma_1 \circ \sigma_2 = \varphi(g)\cdot\varphi(h)$

So φ is a homomorphism $\Rightarrow$ Def. 2 is satisfied.

Now assume Def 2, i.e., a homomorphism

$\varphi : G \longrightarrow P(X)$.

We want to find $\alpha : G \times X \longrightarrow X$ which satisfies the two properties.

Define $\alpha : G \times X \longrightarrow X$ by

$\alpha(g, x) = \varphi(g)(x)$

Note $\varphi(g) \in P(X)$ which is a bijection on X, so $\varphi(g)(x)$ makes sense.

So, $\alpha(e,x) = \varphi(e)(x)$

But φ is a homomorphism $\Rightarrow \varphi(e) = Id_X$

$\Rightarrow \alpha(e,x) = \varphi(e)(x) = Id_X(x) = x$

So property 1) is satisfied.

For $g, h \in X$

$$\alpha(g, \alpha(h,x)) = \alpha(g, \varphi(h)(x)) = \varphi(g)\circ\varphi(h)(x)$$
$$= \varphi(gh)(x)$$
$$= \alpha(gh, x)$$

as φ is a homomorphism. Thus Def. 2 is also satisfied.

$\blacksquare$

So one can work w/ any of the two definition.

Let's see some examples of a group action.

<u>Examples</u>

1. Define $\alpha : G \times X \longrightarrow X$ by

$\alpha(g,x) = x \quad \forall \; g \in X.$

Then for $e \in G$, $\alpha(e,x) = x$

$\alpha(g, \alpha(h,x)) = \alpha(g,x) = x$ and

$\alpha(gh, x) = x \implies \alpha(g, \alpha(h,x)) = \alpha(gh, x)$

So α is a group action.

2. Consider $S_3 = \{ \sigma : \{1,2,3\} \to \{1,2,3\} \mid \sigma \text{ is a bijection} \}$

and $X = \{1,2,3\}$. Define

$\alpha : S_3 \times X \longrightarrow X$ by

$\alpha(\sigma, i) = \sigma(i) \quad , \sigma \in S_3 \; , i \in X$, i.e.,

if $\sigma \in S_3$ is for example $[1\,2\,3]$, then

$\alpha(\sigma, 1) = \sigma(1) = 2$

$\alpha(\sigma, 2) = \sigma(2) = 3$

$\alpha(\sigma, 3) = \sigma(3) = 1$

and similarly for all $\sigma \in S_3$.

Then $\alpha(e, i) = e(i) = i \quad \forall\, i \in \{1, 2, 3\}$

and $\alpha(\sigma, \alpha(\tau, i)) = \alpha(\sigma, \tau(i)) = \sigma \circ \tau(i)$

$$= \alpha(\sigma \circ \tau, i)$$

Thus α is an action of S_3 onto $\{1, 2, 3\}$.

Infact, by the same procedure as above

$$\alpha : S_n \times \{1, 2, \dots, n\} \longrightarrow \{1, 2, \dots, n\}$$

given by $\alpha(\sigma, i) = \sigma(i)$, $\sigma \in S_n$, $i \in \{1, 2, \dots, n\}$

is an action of S_n onto $\{1, 2, \dots, n\}$.

3. Let G be a group and let $X = G$ itself

Define $\alpha : G \times G \to G$ as

$$\alpha(g, h) = g \cdot h = gh$$

Then $\alpha(e, h) = e \cdot h = h$

$$\alpha(g, \alpha(h, k)) = \alpha(g, hk) = g(hk) = (gh)k$$

as G is associative.

Thus α is an action of G onto itself. This is called the left action of G onto itself.

4. Let G be a group and let $X = G$.
Define $\alpha : G \times G \to G$ by

$$\alpha(g, h) = ghg^{-1} \quad \forall\ g, h \in G$$

$$\alpha(e, h) = ehe^{-1} = h$$

$$\alpha(g, \alpha(h, k)) = \alpha(g, hkh^{-1}) = g(hkh^{-1})g^{-1}$$
$$= (gh)k(gh)^{-1}$$

So α is an action of G onto itself called the conjugate action or action by conjugation.

Related to any action of G onto X, we have two sets :-

Orbit of an element in X.

For $x \in X$, the orbit of $x \in X$ is

$$O_x = \{ \alpha(g,x) = g\cdot x \mid g \in G \}$$

i.e. for $x \in X$, look at the action of all elements in G on x and collect them, it will be O_x. Note that $O_x \subseteq X \quad \forall x \in X$.

Stabilizer of an element in X

For $x \in X$, the stabilizer of x is

$$Stab(x) = \{ g \in G \mid \alpha(g,x) = g\cdot x = x \}$$

i.e., it is the set of all those elements in G whose action on x do not move x.

Note $Stab(x) \subseteq G \quad \forall x \in X$.

Note that due to property I in Def^n 1 of an action, $e\cdot x = x \quad \forall x \in X \Rightarrow \forall x \in X$, $e \in Stab(x) \Rightarrow Stab(x) \neq \phi$.

In fact,

Prop:- $Stab(x) \leq G$, $\forall x \in G$.

Proof:- Question on Assignment 5.

Let's calculate $Stab(x)$ for examples 3. and 4. above.

3) Left action of G onto itself.

$$Stab(x) = \{ g \in G \mid \alpha(g, x) = x \}$$
$$= \{ g \in G \mid g \cdot x = x \} = \{e\}$$

So, in this case $\forall x \in G$, $Stab(x) = \{e\}$.

4) Conjugate action of G onto G.

For $a \in G$,

$$Stab(a) = \{ g \in G \mid \alpha(g, a) = a \}$$
$$= \{ g \in G \mid gag^{-1} = a \}$$
$$= \{ g \in G \mid ga = ag \}$$
$$= C(a),$$ the centralizer of a in G.

So, for the conjugate action, the Stabilizer of any $g \in G$ is $C(g)$.

The reason I introduced these objects is that they are intimately related to each other, which is the content of the next theorem.

<u>Theorem</u> (Orbit-Stabilizer Theorem)

Let G act on a set x. Then $\forall x \in X$

$[G : Stab(x)] = |O_x|$. If G is finite, then

since $[G : Stab(x)] = \dfrac{|G|}{|Stab(x)|} \Rightarrow |G| = |Stab(x)| \cdot |O_x|$.

So the theorem is telling us that the # of distinct left or right cosets of $Stab(x)$ in G is precisely the cardinality of O_x.

We'll prove this theorem in the next lecture.